# Continuum-continuum population trapping

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(Received 31 October 1988)

We consider the problem of coherent population trapping in a model quantum system consisting of two (and briefly three) coupled continua of energy levels. While an earlier study [Z. Deng and J. H. Eberly, Phys. Rev. A **34**, 2492 (1986)] has demonstrated population trapping in a closely related system, it has done so only for the case of a broad featureless continuum. We will demonstrate, on the other hand, that the time evolution of the continuum is qualitatively very different when different types of continua are allowed, and that population trapping is not likely to occur *except* in the broad featureless continuum. We do show, however, that in some cases other counterintuitive results surface.

## I. INTRODUCTION

Recently, there has been some interest in the phenomenon of coherent population trapping in quantum systems. Coherent population trapping occurs when the population is confined to some small subset of the entire set of energy levels of the system, even though this subset of energy levels may seem to be strongly coupled to the remainder of the system. Relevent studies include those dealing with discrete systems,[1-7] those dealing with the interaction of discrete levels and continuous bands of energy levels,[8-11] and very recently with entirely continuous systems.[12]

The latter study, of Deng and Eberly (hereafter referred to as DE), is of the most concern to us because it has very recently generated a certain amount of controversy.[13,14] DE investigated population trapping in the system illustrated in Fig. 1. Figure 1 depicts an infinite sequence of *featureless* continua, with transitions allowed between adjacent continua. DE demonstrated that the system population could be almost entirely trapped in the continuum initially containing the population, even if this continuum were very strongly coupled to all of the other continua. In fact, the stronger the coupling, the higher the degree of trapping.

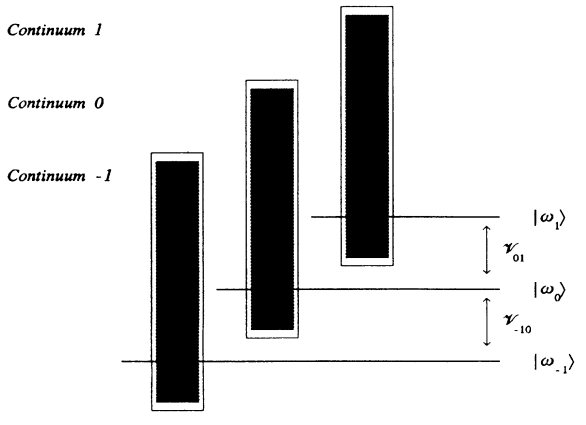


FIG. 1. The system investigated in DE. It has an infinite sequence of featureless continua, with transitions taking place only between adjacent continua.

However, DE leaves a number of interesting points to our imagination. Reference 13 points out two minor flaws in the work of DE: first, it does not actually show that the population is trapped, but only that it manages to somehow return to the initial continuum by $t \to \infty$, and, second, the complication of infinitely many continua (as opposed to just two or three of them) is unnecessary. Both of these minor points will be addressed in passing.

More interesting is the seeming implication (if only by omission of discussion of any other cases) that the broad featureless continuum is somehow representative of *all* continua. Are we to expect that all other continua exhibit population trapping as well? This is the major question we will address in the present study.

Three types of continua will be discussed. These may be categorized as *broad* continua, *narrow* continua, and *strictly limited* continua. While these ideas will be defined more precisely in the following sections, they may be understood roughly as follows: Each continuum is characterized in part by its "width" in frequency space. A continuum is "broad" if its width is much greater than the Rabi frequency, and is "narrow" if its width is less than the Rabi frequency. The difference between a narrow continuum and a "strictly limited" continuum lies in the technical definition of the continuum width. The width of a narrow continuum is defined to be the half-width at half maximum (HWHM) of the coupling coefficients, while the width of a strictly limited continuum is the region outside of which there are no energy levels. For example, a Lorentzian continuum can be either broad or narrow, but cannot be strictly limited since there are energy levels all the way to $\omega \to \pm\infty$.

We will find that these continua evolve qualitatively in very different ways. Like DE, we will mainly concern ourselves with the case of very strong coupling, as in an intense laser field. In this limit, only the broad continuum exhibits the population trapping found by DE. While neither of the other continuum types exhibits population

 

trapping, neither do they exhibit the same qualitative behavior as each other.

## II. THE (CONTINUUM,CONTINUUM) SYSTEM

Consider the system depicted in Fig. 2. In this system, we have two continua interacting with each other through some potential. If the coupling coefficients $V_{01}(\omega_0,\omega_1)$ connecting the two continua are not time dependent, we can try to solve for the time evolution of the system by means of the Laplace transform.

We will suppose that the probability amplitudes of the levels in continuum 0 are denoted by $C_0(\omega_0)$, where $\omega_0$ is some continuous index like the frequency. The energies of these (undriven) levels will be $E_0(\omega_0)$. Similarly, continuum 1 will have probability amplitudes $C_1(\omega_1)$ and energies $E_1(\omega_1)$. (Of course, the probability amplitudes are also functions of time, though not explicitly shown.) Since we will be putting $\hbar=1$, we will simply suppose that $E_n(\omega_n)=\omega_n$. The interaction between the $\omega_0$th level of continuum 0 and the $\omega_1$th level of continuum 1 will be $V_{01}(\omega_0,\omega_1)$. Thus, Schrödinger's equations will be

$$i\dot{C}_0(\omega_0)=\omega_0 C_0(\omega_0)+\int V_{01}(\omega_0,\omega_1)C_1(\omega_1)d\omega_1, \tag{1a}$$

$$i\dot{C}_1(\omega_1)=\omega_1 C_1(\omega_1)+\int V_{10}(\omega_1,\omega_0)C_0(\omega_0)d\omega_0. \tag{1b}$$

The ranges of integration will be specified later when we make a more specific choice of the interaction $V_{01}(\omega_0,\omega_1)$.

Because we are interested in the problem of trapping population in continuum 0, we will specify initial conditions at time $t=0$ of

$$C_0(\omega_0)=\alpha(\omega_0) \tag{2a}$$

and

$$C_1(\omega_1)\equiv 0, \tag{2b}$$

where $\alpha$ is some function at our disposal.

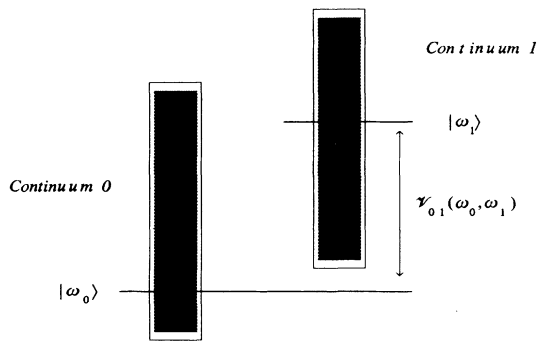


FIG. 2. A system with two continua, labelled 0 and 1. No transitions take place between two levels of the *same* continuum. Inter-continuum transitions are controlled by the function $\mathcal{V}_{01}(\omega_0,\omega_1)$.

For reasons which are made clear in Appendix A, there is no easy way to compute the Laplace transform without some simplifying assumptions about the nature of $V_{01}$. Specifically, let us consider the simple approximation

$$V_{01}(\omega_0,\omega_1)\approx V_0(\omega_0)V_1(\omega_1). \tag{3}$$

For example, for *featureless* continua we have simply $V_{01}(\omega_0,\omega_1)\equiv$const, which is clearly consistent with Eq. (3). It will be clear from Appendix A that there are also more accurate approximations that would allow us to compute the Laplace transform, but we will make no use of them here.

Using the method of Appendix A, the Laplace transforms of $C_0(\omega_0)$ and $C_1(\omega_1)$ are found to be simply

$$\tilde{C}_0(\omega_0)=\frac{1}{s+i\omega_0}\left[\alpha(\omega_0)+\frac{\bar{\alpha}(s)}{1-\chi_0(s)\chi_1(s)}\chi_1(s)V_0(\omega_0)\right], \tag{4}$$

$$\tilde{C}_1(\omega_1)=\frac{\bar{\alpha}(s)}{1-\chi_0(s)\chi_1(s)}\frac{V_1(\omega_1)}{s+i\omega_1}, \tag{5}$$

where

$$\bar{\alpha}(s)=\int\frac{V_0(\omega_0)\alpha(\omega_0)}{is-\omega_0}d\omega_0, \tag{6}$$

and

$$\chi_n(s)=\int\frac{V_n(\omega)^2 d\omega}{is-\omega}. \tag{7}$$

We will use these expressions to discuss both the cases of broad and narrow continua, but will adopt an alternate technique for the case of the strictly limited continuum.

## III. BROAD CONTINUA

Like DE, we will accept the infinite featureless continuum as a prototype broad continuum. In fact, to simplify comparison, we will choose also the same initial conditions:

$$\alpha(\omega_0)=\frac{1}{\omega_0+i\gamma}\left[\frac{\gamma}{\pi}\right]^{1/2}, \tag{8}$$

which describes an initial Lorentzian population distribution of width $\gamma$. The featureless continua are characterized by the equations

$$V_0(\omega_0)\equiv 1 \tag{9a}$$

and

$$V_1(\omega_1)\equiv\mathcal{V}, \tag{9b}$$

where $\mathcal{V}$ is a dimensionless constant governing the strength of the interaction. Under these conditions we find that

$$\bar{\alpha}(s)=\frac{-2}{s+\gamma}\sqrt{\gamma\pi}, \tag{10}$$

with

$$\chi_0(s) \equiv -i\pi \tag{11a}$$

and

$$\chi_1(s) = -i\pi\mathcal{V}^2. \tag{11b}$$

Thus,

$$C_1(\omega_1) = \frac{-2\mathcal{V}}{(1+\pi^2\mathcal{V}^2)(\gamma - i\omega_1)}(e^{-i\omega_1 t} - e^{-\gamma t})\sqrt{\gamma\pi}, \tag{12}$$

and, in the limit $t \to \infty$, the total population of continuum 1 goes to

$$|C_1|^2 \to \frac{4\pi^2\mathcal{V}^2}{(1+\pi^2\mathcal{V}^2)^2}. \tag{13}$$

As in DE, this goes to zero in *both* the limits $\mathcal{V} \to 0$ and $\mathcal{V} \to \infty$, though for certain intermediate interaction strengths significant population can be transferred to continuum 1. Note that we do not *need* to take the limit $t \to \infty$ to observe the population trapping since (by inspection), it is obvious that the total continuum 1 population at *any* finite time never exceeds four times the limiting value.

By the same token, the probability amplitudes in continuum 0 are just

$$C_0(\omega, t) = \frac{\alpha(\omega)}{1+\pi^2\mathcal{V}^2}[2\pi^2\mathcal{V}^2 e^{-\gamma t} + (1-\pi^2\mathcal{V}^2)e^{-i\omega t}]. \tag{14}$$

Thus, for $\mathcal{V} \to \infty$, while the population is trapped in continuum 0, it is not actually trapped in the initial state. As $t$ becomes much larger than $\gamma^{-1}$, the population does indeed return to the initial state, but 180° out of phase from what would have been expected if the population had simply remained in the state.

The case $\mathcal{V} = 1/\pi$ is also of some interest. In this case, all of the population is transferred from continuum 0 to continuum $-1$. At the critical interaction strength $\mathcal{V} = 1/\pi$, the probability amplitudes of the levels in continuum 1 at large times $t$ are

$$C_1(\omega_1) = -i\alpha(\omega_1)e^{-i\omega_1 t}. \tag{15}$$

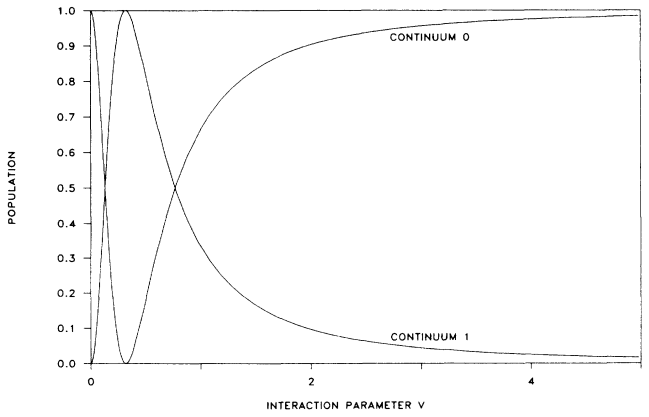


FIG. 3. The steady-state total populations of featureless continua 0 and 1 for various values of the interaction parameter $\mathcal{V}$. The population is initially in continuum 0 and is trapped there for *either* $\mathcal{V} \to 0$ *or* $\mathcal{V} \to \infty$. For $\mathcal{V} = 1/\pi$, the population is entirely transferred to continuum 1.

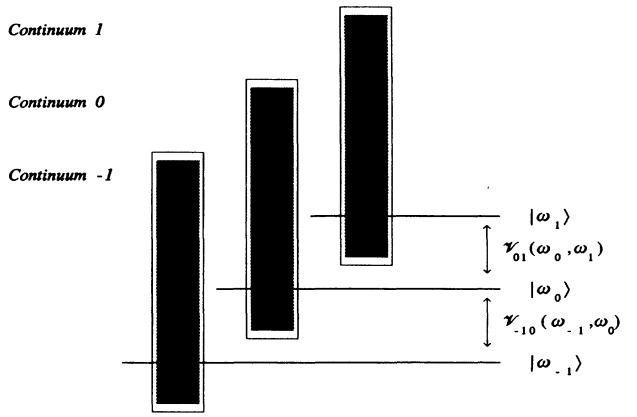


FIG. 4. A system with three continua, labelled $-1$, 0, and 1. Transitions can take place between continua $-1$ and 0, and between continua 0 and 1, but not between continua $-1$ and 1.

Thus, except for a phase shift, it is just as if the population were initially all in continuum 1 and there were no interaction at all. The populations in continua 0 and 1 are shown in Fig. 3 for various values of $\mathcal{V}$.

Incidentally, the same methods can be applied without much difficulty to a system of three interacting continua, as pointed out in Ref. 13. This system is depicted in Fig. 4. We suppose that the interaction between continuum $-1$ and continuum 0 has strength $\mathcal{V}_{-1}$ and that between continuum 0 and continuum 1 has strength $\mathcal{V}_1$. There are two cases of interest. If the initial population is entirely contained in continuum $-1$, then as $t \to \infty$ the following results hold:

$$|C_{-1}|^2 \to \left[\frac{1+\pi^2(\mathcal{V}_1^2 - \mathcal{V}_{-1}^2)}{1+\pi^2(\mathcal{V}_1^2 + \mathcal{V}_{-1}^2)}\right]^2, \tag{16}$$

$$|C_0|^2 \to \left[\frac{2\pi\mathcal{V}_{-1}}{1+\pi^2(\mathcal{V}_1^2 + \mathcal{V}_{-1}^2)}\right]^2, \tag{17}$$

$$|C_1|^2 \to \left[\frac{2\pi^2\mathcal{V}_1\mathcal{V}_{-1}}{1+\pi^2(\mathcal{V}_1^2 + \mathcal{V}_{-1}^2)}\right]^2. \tag{18}$$

For $\mathcal{V}_{-1}$ large or $\mathcal{V}_1$ large, the population of continuum 0 remains small. The population is either trapped in continuum $-1$ or else it is entirely transferred to continuum 1. The latter happens if $\mathcal{V}_{-1} \cong \mathcal{V}_1$, while the former happens if $\mathcal{V}_{-1} \gg \mathcal{V}_1$ *or* $\mathcal{V}_1 \gg \mathcal{V}_{-1}$. This situation is depicted in Fig. 5.

The other case of interest is that of the population initially in continuum 0. Then as $t \to \infty$ we have

$$|C_{-1}|^2 \to \left[\frac{2\pi\mathcal{V}_{-1}}{1+\pi^2(\mathcal{V}_1^2 + \mathcal{V}_{-1}^2)}\right]^2, \tag{19}$$

$$|C_0|^2 \to \left[\frac{1-\pi^2(\mathcal{V}_1^2 + \mathcal{V}_{-1}^2)}{1+\pi^2(\mathcal{V}_1^2 + \mathcal{V}_{-1}^2)}\right]^2, \tag{20}$$

$$|C_1|^2 \to \left[\frac{2\pi\mathcal{V}_1}{1+\pi^2(\mathcal{V}_1^2 + \mathcal{V}_{-1}^2)}\right]^2. \tag{21}$$

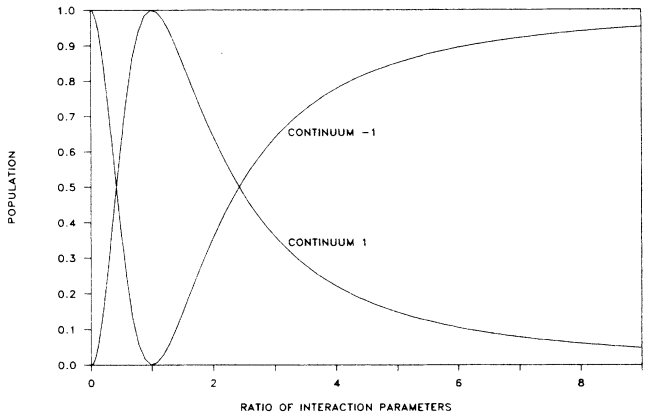


FIG. 5. The steady-state populations in featureless continua $-1$ and 1 for large interaction parameters $\mathcal{V}_{-1}$ and $\mathcal{V}_1$. The population is initially all in continuum $-1$. Continuum 0 contains no population. The distribution of population among continua $-1$ and 1 varies according to the ratio of the interaction parameters, $\mathcal{V}_1/\mathcal{V}_{-1}$. The population is trapped in continuum $-1$ for *either* $\mathcal{V}_1/\mathcal{V}_{-1}\to 0$ *or* $\mathcal{V}_1/\mathcal{V}_{-1}\to\infty$. If $\mathcal{V}_1=\mathcal{V}_{-1}$, the population is entirely transferred to continuum 1.

In this case, the results no longer depend on the relationship between $\mathcal{V}_{-1}$ and $\mathcal{V}_1$. If either of these coefficients is large, the population is trapped in continuum 0.

## IV. NARROW CONTINUA

We will consider the Lorentzian continuum as the prototype of a narrow continuum, but also as an additional example of a broad continuum. To do this, we replace Eq. (9) by the relations

$$V_0(\omega)=V_1(\omega)/\mathcal{V}=\alpha(\omega)=\left[\frac{\sigma/\pi}{\omega^2+\sigma^2}\right]^{1/2}, \tag{22}$$

where $\sigma$ is the "width" of the Lorentzian continua. This $\mathcal{V}$ is slightly different than the interaction strength used earlier, since rather than being dimensionless it has units of frequency. However, it still serves as a basic measure of the interaction strength. Moreover, we have changed the initial conditions of the problem somewhat. From Eqs. (6) and (7),

$$\chi_0(s)=\bar{\alpha}(s)=\chi_1(s)/\mathcal{V}^2=\frac{-i}{s+\sigma}. \tag{23}$$

According to Eq. (5), the Laplace-transformed probability amplitudes in continuum 1 are then

$$\tilde{C}_1(\omega,s)=\frac{i(s+\sigma)}{(s+\sigma)^2+\mathcal{V}^2}\frac{V_1(\omega)}{s+i\omega}. \tag{24}$$

Noting that the roots of the denominator are $-\sigma\pm i\mathcal{V}$ and $-i\omega$, this expression is easily inverted to give the probability amplitudes themselves as

$$C_1(\omega,t)=\frac{-iV_1(\omega)}{(\sigma-i\omega)^2+\mathcal{V}^2}[e^{-\sigma t}\mathcal{V}\sin\mathcal{V}t+(\sigma-i\omega)(e^{-i\omega t}-e^{-\sigma t}\cos\mathcal{V}t)]. \tag{25}$$

The steady-state population of level $\omega$ in continuum 1 is thus

$$|C_1(\omega)|^2_{\rm SS}=\frac{\mathcal{V}^2\sigma/\pi}{(\sigma^2+\omega^2-\mathcal{V}^2)^2+4\sigma^2\mathcal{V}^2}. \tag{26}$$

This function is very small except for $\omega\approx\pm\mathcal{V}$ (for $\sigma\ll\mathcal{V}$). The steady-state population is bunched into two sidebands, far from the center of the continuum. The total population of continuum 1 may also be easily computed as

$$|C_1|^2_{\rm SS}\to\frac{\mathcal{V}^2}{2(\mathcal{V}^2+\sigma^2)}. \tag{27}$$

Since this clearly approaches $\frac{1}{2}$ for large $\mathcal{V}$, the steady-state population is therefore evenly distributed between the two continua when $\mathcal{V}\gg\sigma$.

Thus, we no longer have the population trapping we had in the case of a broad continuum. In fact, we see that for $\mathcal{V}\gg\sigma$ that the system behaves something like a two-level system. The population oscillates rapidly between the two continua. The oscillation is damped at rate $\sigma$, forming a steady-state population distribution with sidebands.

What if $\sigma\gg\mathcal{V}$, so that the Lorentzian is a broad continuum rather than a narrow one? This comparison requires some care in order to match the parameters of the Lorentzian model with the parameters of the featureless model used earlier, since the interpretation of $\mathcal{V}$ has changed. We must replace the $\mathcal{V}$ in our Lorentzian model by $\mathcal{V}\sqrt{\pi\sigma}$. Nevertheless, we still see that the population of continuum 1 does approach zero as $\sigma$ becomes much larger than $\mathcal{V}$. Naturally, we do not expect exact numerical agreement between the Lorentzian model and the featureless model, since the initial conditions still differ.

## V. STRICTLY LIMITED CONTINUA

A strictly limited continuum is characterized by a width $\sigma$, just as the narrow continua considered in Sec. IV. For the narrow continuum, $\sigma$ represented the HWHM of the coupling coefficient functions $V_0(\omega)$ and $V_1(\omega)$. For the strictly limited continuum, on the other

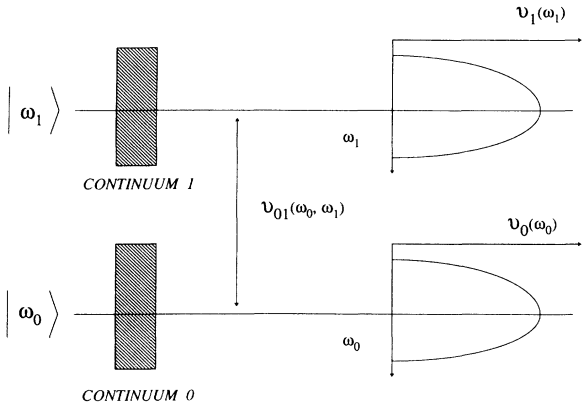


FIG. 6. A (CONTINUUM,CONTINUUM) system with Chebychev continua of the second kind. The coupling coefficients of this band shape go to 0 at $\omega=\pm\sigma$.

hand, we have the stronger condition $|\omega|<\sigma$.

As a prototype for the strictly limited continuum, we will use what we call the *Chebychev continuum of the second kind*, depicted in Fig. 6 and defined by the relation

$$V_0(\omega)=V_1(\omega)/\mathcal{V}=\left[\frac{2}{\pi\sigma}\right]^{1/2}\left[1-\frac{\omega^2}{\sigma^2}\right]^{1/4}. \tag{28}$$

Lest the definition of this band shape seem too novel, it should be noted that all strictly limited continua are, in a sense, qualitatively similar to this band shape.[15] This function square integrates to unity just as the Lorentzian band shape does.

It can be shown[16] that there is a similarity transformation under which the Hamiltonian of this system can be made tridiagonal—i.e., turned into a ladder system. That this can be done quite easily for the (1,CONTINUUM) system has been shown in a previous publication.[17] The same method can be followed for our (CONTINUUM,CONTINUUM) system, and the details are presented in Appendix B.

In Appendix B, we produce a basis with probability amplitudes $a_n(t)$ with $n=\pm1,\pm2,\cdots$. The index $n=0$ is omitted for notational convenience. The indices $n<0$ are mixtures of continuum 0 states, while the indices $n>0$ are mixtures of continuum 1 states. State $n$ in the tridiagonal basis has $n$ nodes (i.e., $n$ values $\omega$ at which the population is zero) when translated back into the conventional continuum basis. We find that in the tridiagonalized basis the Schrödinger equations of motion for the system are

$$i\frac{d}{dt}a_n(t)=\frac{\sigma}{2}a_{n-1}(t)+\frac{\sigma}{2}a_{n+1}(t) \quad \text{for } n<-1, \tag{29}$$

$$i\frac{d}{dt}a_{-1}(t)=\frac{\sigma}{2}a_{-2}(t)+\mathcal{V}a_1(t), \tag{30}$$

$$i\frac{d}{dt}a_1(t)=\mathcal{V}a_{-1}(t)+\frac{\sigma}{2}a_2(t), \tag{31}$$

$$i\frac{d}{dt}a_n(t)=\frac{\sigma}{2}a_{n-1}(t)+\frac{\sigma}{2}a_{n+1}(t) \quad \text{for } n>1. \tag{32}$$

If we could find a complete set of eigenstates for this set of equations, then we could construct any arbitrary solution as a superposition of the eigenstates. There turn out to be two different kinds of eigenstates, which we will denote as $a_n^{(\mathrm{I})}$ and $a_n^{(\mathrm{II})}$. These are given by the (unnormalized) expressions

$$a_n^{(\mathrm{I}\pm)}(t)=e^{\mp i\omega' t}\begin{cases}(\pm\lambda)^{-n} & \text{for } n<0\\ \pm(\pm\lambda)^{n} & \text{for } n>0,\end{cases} \tag{33}$$

$$a_n^{(\mathrm{II})}(t,k)=e^{-i\sigma t\cos k}\begin{cases}-\mathcal{V}^2e^{ik}\sin k(n+1)\\ +\dfrac{\sigma^2}{4}\sin kn & \text{for } n<0\\ -\dfrac{\sigma\mathcal{V}}{2}e^{ikn}\sin k & \text{for } n>0.\end{cases} \tag{34}$$

The auxiliary quantities $\lambda$ and $\omega'$ are defined by

$$\omega'=\mathcal{V}(1+\lambda^2), \tag{35}$$

and

$$\lambda=\frac{\sigma}{2\mathcal{V}}. \tag{36}$$

Thus there are two distinct $a_n^{(\mathrm{I})}$ eigenfunctions, given by $a_n^{(\mathrm{I}+)}$ and $a_n^{(\mathrm{I}-)}$, and infinitely many eigenfunctions $a_n^{(\mathrm{II})}(k)$ depending on the "wave-number" parameter $k$. The $a_n^{(\mathrm{I})}$ eigenstates exist only for $\mathcal{V}>\sigma/2$.

In the interest of simplicity, let us suppose that the initial conditions of the system are given by

$$a_{-1}=1 \quad \text{and} \quad a_{n\neq-1}=0 \quad \text{at } t=0. \tag{37}$$

This means that we are initially placing all of the population into continuum 0, in the least dephased basis state. As we might have expected from the fact that the $a_n^{(\mathrm{I})}$ exist only for $\mathcal{V}>\sigma/2$, there are two regimes of interest. The full solution is indeed a superposition of the eigenfunctions, but with the cases $\mathcal{V}<\sigma/2$ and $\mathcal{V}>\sigma/2$ requiring separate treatment.

The low field-strength case $\mathcal{V}<\sigma/2$ is somewhat the simpler of the two. We find[18] that

$$a_n(t)=\frac{i}{\pi}\int_{-\pi}^{\pi}\frac{e^{ik}}{(\sigma^2/4)-\mathcal{V}^2e^{2ik}}a_n^{(\mathrm{II})}(t,k)\,dk, \tag{38}$$

$$a_n(t)=\begin{cases}i^{n+1}\dfrac{2}{\sigma t}\displaystyle\sum_{m=0}^{\infty}(-1)^m(2m-n)\lambda^{-2m}J_{2m-n}(\sigma t), & n<0\\ (-i)^n\dfrac{2}{\sigma t}\displaystyle\sum_{m=0}^{\infty}(-1)^m(n+2m+1)\lambda^{-2m-1}J_{n+2m+1}(\sigma t), & n>0,\end{cases} \tag{39}$$

where $J_n$ is the $n$th Bessel function.

This solution is already sufficient to investigate the special cases $\lambda=1$ ($\mathcal{V}=\sigma/2$) and $\lambda=\infty$ ($\mathcal{V}=0$). For $\mathcal{V}=\sigma/2$, we find simply

$$a_n(t)=\begin{cases} i^{n+1}J_{-(n+1)}(\sigma t), & n<0 \\ (-i)^n J_n(\sigma t), & n>0, \end{cases} \tag{40}$$

while for $\mathcal{V}=0$, we find

$$a_n(t)=\begin{cases} i^{n+1}\dfrac{2}{\sigma t}(-n)J_{-n}(\sigma t), & n<0 \\ 0, & n>0. \end{cases} \tag{41}$$

Thus, for $\mathcal{V}=\sigma/2$, the population simply divides into two equal wave packets. One wave packet moves up the ladder of continuum 1, while the other moves the ladder of continuum 0. The two continua unrecoverably absorb equal amounts of population. When $\mathcal{V}=0$, on the other hand, continuum 1 naturally receives no population. All of the population is unrecoverably dephased into continuum 0, as the wave packet moves down the ladder.

In the case of larger interactions $\mathcal{V}>\sigma/2$, we find

$$a_n(t)=\frac{8\lambda^2}{2\pi i\sigma^2}\int_{-\pi}^{\pi}\frac{e^{ik}}{1-\lambda^2e^{-2ik}}a_n^{(\mathrm{II})}(t,k)\,dk+\frac{1-\lambda^2}{2\lambda}(a_n^{(I+)}-a_n^{(\mathrm{I}-)}) \tag{42}$$

$$a_n(t)=\begin{cases} \dfrac{2(-i)^{n+1}}{\sigma t}\displaystyle\sum_{m=0}^{\infty}(-1)^m(n+2m+2)\lambda^{2m+2}J_{n+2m+2}(\sigma t)+\frac{1-\lambda^2}{2\lambda}\lambda^{-n}[e^{-i\omega' t}-(-1)^n e^{i\omega' t}], & n<0, \\ \dfrac{2(-i)^n}{\sigma t}\displaystyle\sum_{m=0}^{\infty}(-1)^m(n-2m-1)\lambda^{2m+1}J_{n-2m-1}(\sigma t)+\frac{1-\lambda^2}{2\lambda}\lambda^{n}[e^{-i\omega' t}+(-1)^n e^{i\omega' t}], & n>0. \end{cases} \tag{43}$$

Though these expressions look a trifle complex, they have several easily derived consequences. Clearly, a portion of the population permanently oscillates between the two continua, while another portion unrecoverably dephases into the continua. Since all of the Bessel-function terms decay to zero as $t\to\infty$, the total oscillating (as opposed to decaying) population is seen to be [19]

Total oscillating population

$$=\sum_{n=-\infty}^{\infty}\left|\frac{1-\lambda^2}{2\lambda}(a_n^{(\mathrm{I}+)}-a_n^{(\mathrm{I}-)})\right|^2$$
$$=1-\lambda^2,\ \lambda\le 1. \tag{44}$$

When $\mathcal{V}\gg\sigma$, we have $\lambda\ll 1$, and consequently most of the population *does not* dephase into the continua.

This result is in contrast to the results we saw for the narrow but not strictly limited continua. There, the system behaved like a two-level system, with population oscillating rapidly between the two continua when $\mathcal{V}\gg\sigma$. However, there was *always* some nonzero degree of damping, so that a steady-state distribution was always produced after a time $\sigma^{-1}$, no matter how small $\sigma>0$ became. In the steady-state distribution, 50% of the population went to each continuum, and *no* population oscillated. For the strictly limited continuum, the two-level behavior is present, but there is no damping. Thus, after a time $t>\sigma^{-1}$ the narrow and strictly limited continua are qualitatively very different from each other.

Why does this happen? If we recall the steady-state population distribution produced in the narrow continua systems, it consisted largely of two sidebands at frequency $\omega=\pm\mathcal{V}$. In the strictly limited continuum, there can be no such sidebands since there are no energy levels at those frequencies.

## VI. SUMMARY AND DISCUSSION

We have investigated model quantum systems containing two or three interacting continua, and no discrete levels. In order to carry out some of the calculations involved, we have introduced (in Appendix A) a technique for computing the Laplace transform of systems that have a characteristic we refer to as *low driver rank*. Low driver-rank systems are characterized by a small integer parameter $M$. The smaller the $M$, the easier to compute the Laplace transform.

For broad two-continua systems, we have found that if each of the populations is initially in one of the continua, then the population is permanently trapped in that continuum in *either* the case of weak interactions *or* the case of very strong interactions. For some intermediate interaction strength, the population is almost entirely transferred to the other continuum. For narrow continua, the population initially oscillates between the two continua, and then settles down into a steady-state distribution. If the interaction strength is large, the steady-state population returns entirely to the continuum initially containing it. For strictly limited continua, the situation is the same, except that not all of the population participates in damping. Indeed, for very large interaction

strengths none of the population participates in the damping, and the two-level-like oscillations of the system continue forever.

For broad three-continua systems (with the continua arranged in a ladder), the disposition of population is slightly more complex than in the two-continua systems. If all of the population is initially in the middle continuum, then the population is permanently trapped there for either very weak or very strong interactions. On the other hand, if the population is initially in one of the end continua, for very strong interactions the population can either remain trapped, or else it can be shared with the other end continuum. However, no population moves to the middle continuum.

An interesting extension to this work would be to examine the effects of half-infinite rather than infinite continua—i.e., $\omega_{\min} < \omega < \infty$. However, we leave this for another day.

## ACKNOWLEDGMENTS

Financial support for this work was provided, in part, by the Robert A. Welch Foundation (Grant No. AT-873), the Office of Naval Research with funding from the Strategic Defense Initiative Organization/Office of Innovative Science and Technology, the National Science Foundation (Grant No. PHY-8519293), Cray Research Inc., Convex Computer Corporation, and Texas Advanced Technology and Research Program.

## APPENDIX A: LAPLACE TRANSFORMS IN SYSTEMS OF LOW DRIVER RANK

While each investigator seems to work out his or her own tricks for computing Laplace transforms, there is a simple and routine technique that can be used to get Laplace transforms for most model systems actually solved in the literature. Consider a simple model quantum system with the Hamiltonian operator

$$H = H_0 + V. \tag{A1}$$

We will suppose that there is some convenient representation in which $H_0$ is diagonal, and that the "*matrix*"*rank* [20] of the $V$ operator is some small number

$$M = \text{rank} V. \tag{A2}$$

If so, we say that the system is of *low driver rank*. In this case, the $V$ operator can be decomposed into an operator product of the form

$$V = V_L V_R, \tag{A3}$$

where $V_L$ (considered as a matrix) has column-dimension $M$, and $V_R$ has row dimension $M$. $V_L$ and $V_R$ are not to be confused with projection operators. While they can be expressed in terms of projection operators, in practical terms they are most usefully determined by simple inspection of $V$.

If the basic Schrödinger equation under consideration is

$$i\frac{d}{dt}\psi(t) = H\psi(t), \tag{A4}$$

$$\psi(0) = \psi_0, \tag{A5}$$

then the Laplace-transformed Schrödinger equation is

$$i[s\tilde{\psi}(s) - \psi_0] = H\tilde{\psi}(s). \tag{A6}$$

Substituting from Eqs. (A4) and (A6), and rearranging, we find

$$\tilde{\psi}(s) = (is - H_0)^{-1} V_L \tilde{\phi}(s) + i(is - H_0)^{-1}\psi_0, \tag{A7}$$

where the $M$ vector $\phi(t)$ is defined by the expression

$$\phi(t) = V_R \psi(t). \tag{A8}$$

Multiplying Eq. (A7) by $V$ and rearranging gives

$$\tilde{\phi}(s) = i[I - K(is)]^{-1} V_R (is - H_0)^{-1}\psi_0. \tag{A9}$$

Here, $I$ is the $M$-dimensional identity operator, and the $M \times M$ matrix $K(is)$ is defined by

$$K(is) = V_R (is - H_0)^{-1} V_L. \tag{A10}$$

It may not be obvious from Eq. (A8), but the elements of the $M$ vector $\phi(t)$ are typically the probability amplitudes of any discrete levels, along with the various other quantities resulting from the presence of continuous bands. Thus, if quantities like the probability amplitude of the ground state are the main interest, they can usually be gotten from Eq. (A9) by inspection. If not, the full state vector of the system must be found by substituting Eq. (A9) into Eq. (A7). This gives [21]

$$\tilde{\psi}(s) = i\{I + (is - H_0)^{-1} V_L [I - K(is)]^{-1} V_R\} \times (is - H_0)^{-1}\psi_0 \tag{A11}$$

as our final resulting expression for the Laplace transform.

Equation (A11) is nice because calculating the Laplace transform of even some very complicated systems becomes routine. In the case of our (CONTINUUM, CONTINUUM) system, taking a few liberties with standard matrix notation (in order to include continuous indices), we find by using Eq. (3) that

$$V_L = \begin{bmatrix} \vdots & \vdots \\ 0(\omega_0) & V_0(\omega_0) \\ \vdots & \vdots \\ V_1(\omega_1) & 0(\omega_1) \\ \vdots & \vdots \end{bmatrix} \tag{A12a}$$

and

$$V_R = \begin{bmatrix} \cdots & V_0(\omega_0) & \cdots & 0(\omega_1) & \cdots \\ \cdots & 0(\omega_0) & \cdots & V_1(\omega_1) & \cdots \end{bmatrix}, \tag{A12b}$$

where $0(\omega) \equiv 0$. From Eq. (A9),

$$\tilde{\phi}(s) = \frac{i\bar{\alpha}(s)}{1 - \chi_0(s)\chi_1(s)} \begin{bmatrix} 1 \\ \chi_1(s) \end{bmatrix}, \tag{A13}$$

where $\bar{\alpha}(s)$ and $\chi_n(s)$ are given by Eqs. (6) and (7). Substituting this into Eq. (A7) gives Eqs. (4) and (5) as the full Laplace transform of the system.

## APPENDIX B: LADDER SIMILARITY TRANSFORMATION

For any given Hamiltonian operator, it is possible to find a variety of similarity transformations which convert it to a tridiagonal form, thus producing a ladder system. In some cases it is possible to construct this similarity transformation analytically, following the method of Ref. 17.

In the case of our (CONTINUUM,CONTINUUM) system, the key is to consider the functions $\mathcal{V}_0(\omega)^2$ and $\mathcal{V}_1(\omega)^2$ as "weight functions" for the purposes of defining orthogonal polynomials. Since there are two weight functions, we will construct two sets of orthogonal polynomials $p_n^{(0)}(\omega)$ and $p_n^{(1)}(\omega)$. Actually, it is more convenient to use normalized weight functions $w^{(0)}(\omega)$ and $w^{(1)}(\omega)$, defined by

$$w^{(k)}(\omega)=(\mathcal{V}_k(\omega)/\mathcal{V}^{(k)})^2, \quad k=0,1, \tag{B1}$$

where

$$\mathcal{V}^{(k)}=\left[\int V_k(\omega)^2 d\omega\right]^{1/2}, \quad k=0,1. \tag{B2}$$

Thus

$$\int w^{(k)}(\omega)p_n^{(k)}(\omega)p_m^{(k)}(\omega)d\omega=\delta_{nm}, \quad k=0,1. \tag{B3}$$

We define basis vectors $u_n$ for $n=\pm1,\pm2$,etc., omitting $n=0$, by

$$u_{-n}=\begin{bmatrix}\mathcal{V}_0(\Delta)/\mathcal{V}^{(0)}p_{n-1}^{(0)}(\Delta)\\ 0(\Delta)\end{bmatrix} \tag{B4a}$$

and

$$u_n=\begin{bmatrix}0(\Delta)\\ \mathcal{V}_1(\Delta)/\mathcal{V}^{(1)}p_{n-1}^{(1)}(\Delta)\end{bmatrix}, \tag{B4b}$$

where we define $0(\Delta)\equiv0$.

Computing the matrix elements of the Hamiltonian is now straightforward, and gives

$$H=\begin{bmatrix}
\ddots & & & & & & \\
 & \frac{-a_1^{(0)}}{b_1^{(0)}} & \frac{1}{b_1^{(0)}} & & & & \\
\ddots & \frac{1}{b_1^{(0)}} & \frac{-a_0^{(0)}}{b_0^{(0)}} & \mathcal{V} & & & \\
 & & \mathcal{V} & \frac{-a_0^{(1)}}{b_0^{(1)}} & \frac{1}{b_1^{(1)}} & \ddots & \\
 & & & \frac{1}{b_1^{(1)}} & \frac{-a_1^{(1)}}{b_1^{(1)}} & & \\
 & & & & & \ddots &
\end{bmatrix} \tag{B5}$$

where

$$\mathcal{V}=\mathcal{V}^{(0)}\mathcal{V}^{(1)}. \tag{B6}$$

The $a$ and $b$ constants are defined as the coefficients appearing in the recurrence relation [22]

$$p_{n+1}^{(k)}(x)=(a_n^{(k)}+xb_n^{(k)})p_n^{(k)}(x)-c_n^{(k)}p_{n-1}^{(k)}(x), \quad k=0,1. \tag{B7}$$

For any *known* orthogonal polynomials, these recurrence coefficients can simply be looked up, for example, in Ref. 23. Of course, for a realistic system this would not be true.

For the specific bandshape specified by Eq. (28), the polynomials turn out to be Chebychev polynomials of the second kind, with the resulting coefficients

$$-a_n^{(m)}/b_n^{(m)}=0, \tag{B8}$$

$$1/b_n^{(m)}=\sigma/2. \tag{B9}$$

This leads directly to the tridiagonal Schrödinger equation of Eqs. (29)–(32).


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